FIRST-ORDER DIFFERENTIAL EQUATIONS

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Calculus is the mathematics of change; therefore many real-world events can be modeled using its techniques. To begin, we take the humble **differential equation,** the expression of a rate of change as a function *f* of independent variables *x* and *y*. We model this rate like so:

$$\frac{dy}{dx} = f(x, y)$$

You may be thinking, "Hold on, this sounds like math!" I've been getting that reaction a lot lately, but I'm going to ask you to bear with me, because I believe there is a connection here, a truth to be got at that needs all the signs we know to render. That is to say, language is inadequate, the sum of all words does not equal truth:

$$\sum_{aardvark}^{zwieback} \text{words} \neq \text{truth}$$

The purposes of studying differential equations are various, boiled down to three for our purposes:

> 1. To model such real-world phenomena that require the logic of numbers, their lack of emotion, to process: i.e., the rate at which cancer cells multiply in the lymphatic system of a 68-year-old man, the limiting velocity a son can achieve on a midnight trip to the county hospital without being stopped by the authorities, the concentration of grief in the bloodstream over time.

2. To find solutions and approximations based on these models, helping to telegraph closure, to estimate the impact of daily events—because that's what deaths are, ultimately—on our lives.

3. In essence, to predict the future. You'll see what I mean.

Example 1: Everyone's Father Dies. Imagine it's mid-summer. Imagine that you have just driven home from a night spent at your father's bedside, fueled only by vending machine detritus and bad coffee for fifteen hours, unable to sleep on a slippery vinyl chair. Imagine that the hospital calls just as you are about to step in the shower. It's the news you've been waiting for. For simplicity's sake, we will say that your desolation is at its highest peak at this revelatory moment; we will assume the axiom 'time heals all wounds' to be true. If your grief (g) declines in proportion to time (t) according to some constant (k), express this relationship using a differential equation.

$$\textit{Solution: } \frac{dg}{dt} = kt$$

But, perhaps this is unclear. You've grown unused to math. "Stop!" you're thinking, "all this means nothing to me! I can hardly use a tip calculator!" Don't give up yet; don't put down this paper; don't walk away to make tea or check the score on that game. Stay on that commode. I'll back up a few steps.

See, a **derivative** is simply a method of modeling change. We can start out by thinking of it in small increments: I receive a certain number of books from my father every year of my life (always biography, if you're interested). Now, we can show the time rate of change of books like this:

$$\frac{\Delta\, books}{\Delta\, time} \quad ; \quad b(t) = \text{number of books at time } t$$

Do you follow? So, the goal of the derivative is to pin down the rate at which I am receiving books at any given time t. Sure, I could just call my older sister and ask her what things looked like in that regard in the fall of 1997, but this way sounds a bit more impressive at cocktail parties. Trust me. Moving along—imagine now that in order to find this specific rate, we bring in an arbitrary constant *h* such that:

$$\frac{b(t+h)-b(t)}{h} = \text{the difference in books from time } t \text{ to time } t+h, \text{ divided by } h = \frac{\Delta b}{\Delta t}$$

We're almost through the woods now; breathe deeply, and think of how many people you'll be able to impress with your ability to calculate. Anyway, it's simple: all I'm saying is, if my father gave me 18 books in 1996 and 24 books in 1998, then not only is this example making him out to be more generous than he actually was, but:

$$\frac{\Delta b}{\Delta t} = \frac{b(t+h)-b(t)}{h} = \frac{24-18}{2} = 3 \text{ books per year}$$

One more step. To get the true derivative of *b(t)*, we want *h* to be as small as possible; the smaller it gets, the closer to the exact rate at time *t* we are. So for the win:

$$\frac{db}{dt} = \lim_{h \to 0} \frac{b(t+h)-b(t)}{h}$$

As my father would say, "You savvy that?"

The first example I gave you had its flaws; it was a little simplistic. You were probably thinking that anybody who knows even a molecule about grieving knows that time-healing-wounds thing is bullshit. You're thinking, there are good days and bad days, but it's deluded to think there's some guarantee that my sadness is decreasing constantly and each day will be better than the last until I'm perfectly happy again. That's nonsense. You're right, you're right, of course you're right. Let's try it again.

Example 2: Everyone's Father Dies, Part Two. Imagine that Jeremy—we'll use the name because it's convenient—has just signed a check over to the funeral home for his father's service, for more money than he technically has in his account at First Federal. He thinks how this act shouldn't surprise him, how much of an oversight it was for him not to plan for his father's death, as if he blithely assumed his father would live forever, even after the treatments failed. Imagine that he gets in the car with his sister, Liz, who wants to go for a drink. Jeremy thinks this idea is a bad one, but he agrees, because he can see the day has been rough on Liz, perhaps even worse than it has been for him.

Imagine that at the bar Liz lights a cigarette, even though Jeremy has stopped smoking six months before, encouraged by hospital smoking bans. Liz is blind, so as you imagine her lighting the cigarette, imagine great beauty and elegance, like that of a card shark or a heart surgeon.

"I'll write you a check for half, tomorrow," Liz says, exhaling smoke in a steep plume that joins the general fog coating the bar's ceiling.

"No hurry," says Jeremy.

Imagine that they sit and drink for some time, and that Jeremy is surprised to find himself drunk, and furious with his sister. He thinks she has no idea, of course, because she can't see the furrow in his brow or how tightly he holds his glass or the way his lips are worked up. He watches her as she stubs out her cigarette on the very edge of the metal ashtray. He notices, for the first time, the feathered lines that have appeared at the meeting places of her upper and lower lips.

"Why are you angry?" she asks.

Imagine Jeremy says nothing, in surprise, then says everything. You know this story, I know you do: he's sad, he's bitter for having all the responsibility, she's convenient to blame. He hates himself for blaming her, because she can't help her limitations, then hates himself for using her blindness to excuse her. You and I, we know how this goes, we don't need to revisit it here.

So, if this moment marks Jeremy's turn toward recovery, his first foray into an emotion other than sadness, we can assume again the faulty model that as time (t) passes, grief (g) declines until the general level of human malaise (m) is reached, which we will call the **equilibrium solution.** Model this relationship using a differential equation.

$$\textit{Solution: } \frac{dg}{dt} = k(m - g)$$

Remark: The solution to this equation is the function:

$$g(t) = m - Ce^{-kt}$$

Do you see then, that no matter what amount of grief (g) Jeremy begins with, as time goes on ($t \to \infty$), things always come back to m? This is why m is called the **equilibrium solution.**

Okay, you're saying, I've got this. I want to know more, I want to know how to solve these suckers, how to really wrangle g into a strict relationship with t so I know when to expect happiness, or at least normality, again. I'm with you. Let's talk about **general solutions** for differential equations.

Using the equation established in Example 1,

$$\frac{dg}{dt} = kt$$

we can see that simple integration will produce a general solution rather easily. First, we move things around:

$$dg = kt\,dt$$

Then, we integrate both sides:

$$\int dg = \int kt\,dt$$

Lastly, we tidy up:

$$g = \frac{k}{2}t^2 + C$$

This assumes some basic knowledge of integration on your part, of course, but what did you think this was, a bedtime story? You knew what I meant when I said "detritus" earlier, or if you didn't I hope that you looked it up—are you balking now? If I get this stuff, so can you. Dust off that textbook that's hanging around from college. Ask your son or daughter or roommate or neighbor to explain it to you. Think about it as you did about "detritus;" remember that learning things is how you stay young, it's how you keep your brain healthy, it's how you don't end up in a depressing old age home that smells like band-aids and urine and no one, not even the nurses, remembers your name without the sign on your door.

Example 3: Hot Dish. Imagine (or don't, because at this point who am I kidding?) that a number of casseroles (*C*) are being delivered at some rate to Liz and Jeremy's father's home. Or rather, their late father's home, but perhaps it's not even that anymore; Jeremy has a missed call from the Coldwell Banker agent on his cell phone that he hasn't yet returned. The house could already be in escrow, its dusty carpets and sunny cement patio in transition to the hands of strangers. Although it's Jeremy and Liz's childhood home, they're too busy resenting their father's packrat habits to care much about its impending sale. Liz has taken it upon herself to pack the china; she can tell which pieces are worth keeping by their weight and the raised patterns on their edges. Once in a while she holds up a mug or mixing bowl and asks Jeremy if it has sentimental value, and he usually tells her no. He is glad she has taken on this task, because he is clumsy and would likely break as many pieces as he wrapped in old newspaper.
During all of this the casseroles keep arriving, some announced by the ring of the doorbell, others abandoned on the mat with a flowered card taped to the dish. They have been arriving for a week, their contents varying but nearly all of them bound together with canned soup. Green beans. Egg noodles. Bottled pasta sauce. Curiously square chunks of chicken. Beets, even. Jeremy wonders that people have turned on their ovens in the brutal July heat.

Liz and Jeremy have already cleaned out the attic, which was the major obstacle to the sale of the house, packed as it was with old clothing and photographs and furniture all smoldering in the hot darkness. The boxes of photos are sitting in the trunk of Jeremy's car now, waiting for the trip back to Chicago, where they'll go into his storage locker in his apartment building and continue accumulating dust, the people pictured slowly becoming nameless icons of the past.

"It's going to rain," says Liz, nestling another plate inside a wooden crate. "Do you smell it?"

The guide dog at her feet whines, and Jeremy walks over to the screen door and looks out at the hot, green foliage. Even with all the windows in the house open, there isn't a hint of a breeze, and this realization makes him sweat more, his brow dripping and his shirt drenched. Liz comes to stand beside him, and above their heads, a thunderclap seems to shake the house to its foundation. Jeremy imagines them running out onto the patio as the rain begins, standing arms outstretched, looking up into the downpour, as he's seen in some movies. The rain would streak their faces and camouflage tears, sweat. He thinks this soaking would be therapeutic, but he doesn't move to unlatch the door when the first drops darken the pavement.

It's Liz who opens the door, once the rain has begun in earnest, and scouts out the lasagna that's been sitting next to the potted geraniums for most of the morning. She picks it up, and she's surprised when the water that's collected on the aluminum foil covering splashes all down the front of her cotton dress. She's so surprised by the flume that she drops the dish, a white ceramic one that breaks into several large pieces when it hits the concrete. It appears to happen in silence, with the din of the rain on all the hard surfaces of the house overwhelming the crash.

"Don't move," say Jeremy, for Liz is barefoot, the thin, white skin of her feet smeared with tomato sauce and ricotta.

As his sister stands immobile, Jeremy gets down on his knees and gathers up the enameled shards, sheets of pasta slithering through his fingers, the mist of refracting raindrops in his eyes. Liz stays rooted to the spot as he goes back into the house and returns with a towel. He lifts each of his sister's feet and wipes them clean. They're both soaked through now, but it's not the experience Jeremy thought it would be; he doesn't feel as though his grief has been disguised. When he looks up, Liz is crying, but you probably knew that was coming. You're smart enough to predict that, if you've stuck around for this long.

If we don't count the casualty of the storm, Liz and Jeremy consume $4C^2$ servings of casseroles daily, a function of the number of servings (C) there are in the refrigerator on any given day. More servings of casserole ($16C$) arrive daily, depending on the number of ceramic dishes visible on the house's porch. Therefore the rate of servings in the house at any time t is equal to the rate of incoming servings minus the rate of outgoing (consumed) servings:

$$\frac{dC}{dt} = 16C - 4C^2$$

Solve this equation for a particular solution if Liz and Jeremy have two servings of casserole in the refrigerator on the day their father dies (C=2 at t=0).

Solution:

First, we dry our tears and factor the right side of the equation:

$$\frac{dC}{dt} = 4C(4 - C)$$

Then, we grow angry and consolidate the variables on each side:

$$\frac{dC}{C(4 - C)} = 4\,dt$$

Now, we fall into despair and integrate:

$$\int \frac{dC}{C(4 - C)} = \int 4\,dt$$

To simplify the left side, we create partial fractions, with indifference:

$$\frac{1}{4}\int\left(\frac{1}{C} + \frac{1}{4 - C}\right)dC = \int 4\,dt$$

Accepting our loss, we follow the rules of integration,

$$\ln(C) - \ln(4 - C) = 16t + k$$

which lead us to

$$\frac{C}{4 - C} = ke^{16t}$$

a general solution:

$$C = \frac{4}{1 + ke^{-16t}}$$

Then, since $C(0)=2$, we know by substitution that $k=1/2$, so the **particular solution** is:

$$C = \frac{4}{1 + \frac{1}{2}e^{-16t}}$$

Remark: Notice that as time goes on ($t \to \infty$), the expression $\frac{1}{2}e^{-16t}$ becomes arbitrarily small, so small that we can ignore it altogether and assume $C=4$. This number (4) is sometimes called the **carrying capacity.** So, we might say that the carrying capacity for casserole in Jeremy and Liz's father's house is four servings; that is, as time passes into eternity there will forever be four portions of congealed mushroom soup in enameled dishes, relegated to the bottom shelf of the refrigerator, behind the condiments, waiting to be eaten by the bereaved—at least, according to our model, which we may as well admit does not reflect reality with complete accuracy.

I'd like to make one final point, to introduce one last method of representation, our concluding foray into predicting what will happen beyond the here and now. Say we use the number of tears shed (T) as an approximation of the sadness felt at the death of a loved one. A faulty model, you're saying, and I agree, but we'll assume it in this case anyway, because I'm running out of ways to try to explain this to you.

Example 4: The Airport. Jeremy and Liz are saying goodbye to each other at the airport, they're adjourning to their own lives, they've past security and they're at Liz's gate and Jeremy is about to hug his sister and put her in a seat near the flight attendant's station so that she gets priority boarding without any trouble.

"You'll call about Christmas?" Liz asks, and Jeremy wonders why it is a question until he remembers: they're the only family that's left, now.

"Yeah," he says. "I'll come out to New York."

Jeremy feels guilt for a moment that he lives so far from his sister when she is really all he has, and he feels fear that she is all he has, and sadness that she is all he has. Then he gets everything under control again and embraces Liz awkwardly and walks away to find his own gate and sits, looking ahead, waiting to board.

Later, though, he is surprised to find himself crying in an airplane lavatory at 35,000 feet, really sobbing, letting himself go in a way we shouldn't be comfortable watching. He's missing his father, really missing him, his father who used to explain about thrust and lift and flight trajectories on every family vacation, ad nauseum. Who used to win in-flight trivia contests by calculating the exact location of the midpoint of the flight. Who was endlessly fascinated by the derivative relationships of acceleration and velocity and position. Who, Jeremy has realized, could probably have calculated the exact second of his death, without even the benefit of a calculator, if he had wanted, if he had only had that information about rates of growth of non-Hodgkin lymphoma.

Jeremy's crying, so we'll look away, we'll take what empirical evidence we can and leave him to what consolation he can find in single-ply toilet paper and toy bottles of liquor.

If, then, Jeremy's average number of tears shed in a day is represented by (T) and we wish to find the time t at which the tears will stop—at which he will ostensibly have overcome his loss—then we can begin with a given model of the rate of these tears using a differential equation:

$$\frac{dT}{dt} = \frac{6t}{t^2+1} - \frac{3t}{t^2+1}T$$

That looks pretty complicated, doesn't it? Well, it's going to be, from here on out. We're buckling down now. I'm getting out of the way. I'm not holding your hand anymore.

Solution:

$$\frac{dT}{dt} + \frac{3t}{t^2+1}T = \frac{6t}{t^2+1}$$

Integrating factor: $$\mathrm{r}(t) = e^{\int \frac{3t}{t^2+1}dt} = \left(t^2+1\right)^{3/2}$$

$$\left(t^2+1\right)^{3/2}\frac{dT}{dt} + 3t\left(t^2+1\right)^{1/2}T = 6t\left(t^2+1\right)^{1/2}$$

$$\frac{d}{dt}\left[\left(t^2+1\right)^{3/2}T\right] = 6t\left(t^2+1\right)^{1/2}$$

$$\left(t^2+1\right)^{3/2}T = \int 6t\left(t^2+1\right)^{1/2}dt = 2\left(t^2+1\right)^{3/2} + C$$

$$T = 2 + C\left(t^2+1\right)^{-3/2}$$

if $T(0)=367$,

$$T = 2 + 365\left(t^2+1\right)^{-3/2}$$

Then, when is $T=0$?

$$0 = 2 + 365\left(t^2+1\right)^{-3/2}$$

$$t \approx 22.74 \text{ years}$$

Remark: If Jeremy was born in February of the year 1980 and his father dies in 2008, this moment of oblivion will occur in early September of 2030. Jeremy will be 50 years old.

End Note: Mathematical modeling is an inexact science, as I'm sure you're noticed. All I've tried to do here is to reduce things to their most basic expressions, to provide hope that things can change at understandable, expressible rates. But perhaps that's reductive. Perhaps that's wrong. Perhaps I should have written you a poem instead, or drawn you a picture, or composed a plaintive melody.

Perhaps I should have told you a story. A story that begins when I am born. I am born, and my mother leaves, and my father is left alone to care for me, and for my sister.

I am two, and I drink out of the dog's water dish, on my hands and knees.

I am six, and I hit another boy in my kindergarten class, and the skin on my knuckles splits apart on his slick teeth.

I am thirteen, and I sneak into a movie I haven't paid for, alone. When it's over I call my sister and confess to her, but when she asks, I can't tell her what the film was about.

I am eighteen, and I'm taking my first calculus class. I sit beside a red-haired girl who sleeps, beautifully, peacefully, through every lecture. I am surprised on the day of the final to realize that she has brown eyes, and a low, pleasant voice, and the rapt attention of everyone in the class, including the professor.

I am twenty-three. I buy my first apartment, with the money from my first job, and I live in it, alone. At times it surprises me that the apartment is dark and empty when I return to it at night.

I am twenty-seven, and I come home for a while to watch my father die.

I am ten, again. My sister is in the hospital.

"Lizzie's better?" I ask, when the doctor leaves and my father comes out into the hallway to sit with me, near the nurse's station.

My father doesn't say anything for a while, and then asks me to close my eyes, to imagine twenty-seven cents. I squirm a little, but I do it—we often play games that begin this way. Sightless, the smell of the hospital becomes brazen, aggressive. Rubber shoe soles against the floor make the hallway sound like the inside of a gymnasium.

"Which coins?" my father asks, and for a moment I don't understand his question, but then, I do.

"A quarter and two pennies," I say, proudly.

"Keep them shut," he says. "Imagine there's a nickel in there."

"Just one?" I say.

It's easy: a nickel, two dimes, two pennies. They're bright, in my mind, against a black background.

"Imagine there's twice as many nickels as dimes," he says.

Still easy: two nickels, a dime, seven pennies.

"Imagine there's fifteen coins, total," he says.

I shuffle through the coins in my mind, feeling the corrugated edges of the dime, the surprising heft of the nickel. Behind my eyelids all pennies are newly minted discs, shining, loved. They flow together in a metallic stream, in stacks that appear and disappear according to my will. Everything outside of this problem falls away as I place all of my energy on it, waiting for a weakness to give way, to reveal the answer to me. I am not reasoning so much as feeling, my entire body tensing, the solution dancing just beyond the circumference of my developing mind. I am ten, and my feet do not yet reach the linoleum of the hospital floor, and my hair needs cutting, and my father is the tallest man who has ever lived.

When I open my eyes, my father is gone. In his place, on the welted vinyl that still bears the imprints of his thighs, there are coins.

Three nickels, twelve pennies.